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Conformal theory of spin correlations in the semi-infinite 3-state Potts and self-dual Z_N models

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Abstract. In the conformal theory the magnetization operator of the critical 3-state Potts or self-dual Z_3 model is degenerate at level 6. Solving a sixth-order differential equation, we calculate the bulk 4-spin correlation function and the spin-spin correlation function in the half space. The spin-spin correlation function of the semi-infinite self-dual Z_N model is also obtained for arbitrary N. Both free- and fixed-spin boundary conditions are considered.

With the conformal-invariance approach [1, 2] the critical indices of a large class of two-dimensional critical systems, known as minimal or degenerate theories, have been derived. The conformal theory also provides linear differential equations that determine the many-point correlation functions of these systems.

Cardy [3] has extended the conformal-invariance approach to semi-infinite critical systems with a conformally invariant boundary condition. He showed that the n-point correlation function in the semi-infinite geometry is determined by the same differential equations as the bulk $2n$-point correlation function.

In this paper we first consider the critical 3-state Potts or self-dual Z_3 model. Dotsenko [4] has classified the primary operators of this model according to the conformal theory and calculated the bulk 4-point correlations $\langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle$ and $\langle\varepsilon\varepsilon\sigma\bar{\sigma}\rangle$ of the energy density ε and spin or magnetization operator σ. Since the energy density is degenerate at level 2, these quantities are determined by second-order differential equations. Recognizing that the energy density remains degenerate at level 2 for $Q \neq 3$, Dotsenko and Fateev [5] obtained these 4-point functions for general Q. The differential equations for the bulk 4-point correlations of the energy density also determine the energy-energy correlation function in the half-space, which has been calculated by Cardy [3].

The scaling index of the spin or magnetization operator of the Q-state Potts model has been identified [4, 5] with $\Delta_{p,q}$, $p = \frac{1}{2}(m-1)$, $q = \frac{1}{2}(m+1)$ in the conformal theory, where $\sqrt{Q} = 2\cos[\pi(m+1)^{-1}]$. For the 3-state Potts model $m = 5$ and $pq = 6$. Thus the 4-spin correlation function is determined by a sixth-order differential equation. Despite the high order it can be solved analytically, as we show in the paper. From the solutions we obtain the bulk 4-spin correlation function and the spin-spin correlation functions in the half space for free- and fixed-spin boundary conditions. The Potts spin operator is a vector, and its half-space correlation functions are of interest in the general theory of surface critical phenomena [6]. Previous calculations with conformal invariance [3, 7] of correlations near boundaries have only considered scalar operators.

With an entirely different approach, based on the algebra of parafermion currents, Zamolodchikov and Fateev [8] have derived the bulk 4-spin correlations of critical Z_N-symmetric self-dual systems. In the limit $N \to 3$, corresponding to the 3-state Potts model, their results agree with ours. The 4-spin correlation function only involves two of the six solutions to the sixth-order differential equation, and it is interesting to see what physical considerations rule out the other solutions. Making use of the results of Zamolodchikov and Fateev, we also obtain the spin-spin correlation function of the semi-infinite self-dual Z_N model for arbitrary N.

We assign the spin variable σ the values $\sigma = 1, \exp(2\pi i/3), \exp(4\pi i/3)$ in the three states of the Potts model. The complex conjugate $\bar\sigma$ satisfies $\bar\sigma = \sigma^{-1}$. In the absence of symmetry breaking the correlation functions $\langle\sigma\sigma\rangle$, $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\sigma\bar\sigma\rangle$ and their complex conjugates vanish identically, whereas $\langle\sigma\bar\sigma\rangle$ and $\langle\sigma\sigma\bar\sigma\bar\sigma\rangle$ do not. As mentioned above, the scaling index $\Delta_{2,3} = \frac{1}{15}$ of the spin variable corresponds to degeneracy at level 6. Thus, according to the conformal theory [1, 2] the bulk 4-spin correlation function $G(z_1, \ldots, z_4) = \langle\sigma(1)\bar\sigma(2)\sigma(3)\bar\sigma(4)\rangle$ is annihilated by a sixth-order differential operator. Determining the numerical coefficients with the Virasoro commutation relations, we obtain the partial differential equation

$$\begin{aligned}[\mathscr{L}_{-1}^6 - \tfrac{154}{15}\mathscr{L}_{-1}^4\mathscr{L}_{-2} + \tfrac{3472}{225}\mathscr{L}_{-1}^2\mathscr{L}_{-2}^2 + \tfrac{2254}{45}\mathscr{L}_{-1}^3\mathscr{L}_{-3} - \tfrac{2048}{375}\mathscr{L}_{-2}^3 - \tfrac{39272}{675}\mathscr{L}_{-1}\mathscr{L}_{-2}\mathscr{L}_{-3} - \tfrac{3004}{15}\mathscr{L}_{-1}^2\mathscr{L}_{-4} \\ + \tfrac{38992}{2025}\mathscr{L}_{-3}^2 + \tfrac{10432}{135}\mathscr{L}_{-2}\mathscr{L}_{-4} + \tfrac{385468}{675}\mathscr{L}_{-1}\mathscr{L}_{-5} - \tfrac{547424}{675}\mathscr{L}_{-6}] \times G(z_1, \ldots, z_4) \\ = 0 \qquad (1)\end{aligned}$$

where

$$\mathscr{L}_{-1} = \frac{\partial}{\partial z_1} \qquad \mathscr{L}_{-n} = (-1)^n \sum_{\alpha=2}^{4} \left(\frac{1}{15}(n-1) z_{1\alpha}^{-n} + z_{1\alpha}^{1-n} \frac{\partial}{\partial z_\alpha}\right). \qquad (2)$$

The z_α are complex position coordinates, and $z_{\alpha\beta} = z_\alpha - z_\beta$.

Invariance of the 4-spin correlation function under Möbius mappings of the complex plane onto itself implies the functional form [1, 2]

$$G(z_1, \ldots, z_4) = (z_{12}z_{34})^{-2/15}\psi(x) \qquad x = \frac{z_{13}z_{24}}{z_{14}z_{23}}. \qquad (3)$$

With the help of a computer program for symbolic manipulations [9], we inserted equations (2) and (3) into (1) and obtained the ordinary sixth-order differential equation

$$\begin{aligned}\Big[x^6(x-1)^5 \frac{d^6}{dx^6} + \frac{2}{15}(148x - 77)x^5(x-1)^4 \frac{d^5}{dx^5} \\ + \frac{2}{15^2}(13324x^2 - 13839x + 2674)x^4(x-1)^3 \frac{d^4}{dx^4} \\ + \frac{2}{15^3}(416926x^3 - 647785x^2 + 245555x - 10394)x^3(x-1)^2 \frac{d^3}{dx^3} \\ + \frac{4}{15^4}(1910564x^4 - 3942607x^3 + 2169451x^2 \\ - 122962x - 47656)x^2(x-1) \frac{d^2}{dx^2}\end{aligned}$$

$$+\frac{4}{15^5}(2613146x^5-6709623x^4+4495417x^3$$

$$+191602x^2-523968x-38224)x\frac{\mathrm{d}}{\mathrm{d}x}$$

$$\left.+\frac{32}{15^6}(1634x^4+3144x^3+10019x^2+3144x+1634)(x-1)\right]\psi(x)=0 \qquad (4)$$

for the function $\psi(x)$. Equation (4) has six linearly independent power-series solutions

$$\begin{aligned}
\psi_1(x)&=x^{-2/15}F(\tfrac{2}{5},-\tfrac{1}{5};\tfrac{3}{5};x)\\
\psi_2(x)&=x^{-1/15}F(\tfrac{1}{5},-\tfrac{1}{5};\tfrac{2}{5};x)\\
\psi_3(x)&=x^{4/15}F(\tfrac{4}{5},\tfrac{1}{5};\tfrac{7}{5};x)\\
\psi_4(x)&=x^{8/15}F(\tfrac{4}{5},\tfrac{2}{5};\tfrac{8}{5};x)\\
\psi_5(x)&=x^{-2/15}(1-x)^{1/15}F(\tfrac{1}{5},-\tfrac{1}{5};\tfrac{3}{5};x)\\
\psi_6(x)&=x^{4/15}(1-x)^{1/15}F(\tfrac{1}{5},\tfrac{3}{5};\tfrac{7}{5};x).
\end{aligned} \qquad (5)$$

Here the $F(a, b; c; x)$ are standard hypergeometric functions [10]. Note that $\psi_1(x)$ and $\psi_5(x)$ both vary as $x^{-2/15}$ and that $\psi_3(x)$ and $\psi_6(x)$ both vary as $x^{4/15}$ for $|x|\ll 1$.

Below we will need the continuation relations

$$\psi_i(x)=\left(\frac{1-x}{x}\right)^{2/15}\sum_{j=1}^{6}\alpha_{ij}\psi_j(1-x). \qquad (6)$$

The matrix of coefficients α_{ij} is given by

$$\alpha=\alpha^{-1}=\begin{pmatrix}
a & 0 & b & 0 & 0 & 0\\
0 & 0 & 0 & 0 & a/2b & a/2\\
a/b & 0 & -a & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 3a & -3b\\
0 & 2b & 0 & a/3 & 0 & 0\\
0 & 2a & 0 & -a/3b & 0 & 0
\end{pmatrix} \qquad (7)$$

$$a=\frac{\Gamma(\frac{2}{5})\Gamma(\frac{3}{5})}{\Gamma(\frac{1}{5})\Gamma(\frac{4}{5})}=\frac{1}{2}(\sqrt{5}-1) \qquad b=\frac{1}{2}\frac{\Gamma(\frac{3}{5})^2}{\Gamma(\frac{2}{5})\Gamma(\frac{4}{5})}.$$

Equations (6) and (7) follow from (5) and a standard relation [10] between hypergeometric functions with arguments x and $1-x$.

The bulk 4-spin correlation function, which satisfies similar differential equations in the complex coordinates z_α and $\bar{z}_\alpha$, has the form

$$\begin{aligned}
\langle\sigma(1)\bar{\sigma}(2)\sigma(3)\bar{\sigma}(4)\rangle&=|z_{12}z_{34}|^{-4/15}\sum_{i,j}A_{ij}\psi_i(x)\psi_j(\bar{x})\\
&=\left|\frac{1-x^{-1}}{z_{12}z_{34}}\right|^{4/15}\sum_{i,j}B_{ij}\psi_i(1-x)\psi_j(1-\bar{x})
\end{aligned} \qquad (8)$$

where

$$B_{kl}=\sum_{i,j}A_{ij}\alpha_{ik}\alpha_{jl} \qquad (9)$$

as follows from equation (6). We now determine the coefficients A_{ij}, B_{ij}.

The requirement that the 4-spin correlation function be single-valued when continued around the singular points $x=0$ and 1 rules out all off-diagonal matrix elements except $A_{15}, A_{51}, A_{36}, A_{63}$ and $B_{15}, B_{51}, B_{36}, B_{63}$. However, non-vanishing values for these off-diagonal matrix elements are incompatible with equations (7) and (9). Thus the matrices A_{ij} and B_{ij} are both diagonal.

The 4-spin correlation function of equation (8) must clearly be invariant under interchange of points 1 and 3 or points 2 and 4. This and the diagonal property of A_{ij} imply

$$\sum_i A_{ii}|\psi_i(x)|^2 = |1-x|^{4/15}\sum_i A_{ii}\left|\psi_i\left(\frac{x}{x-1}\right)\right|^2. \tag{10}$$

The relation

$$\psi_i(x) = (1-x)^{2/15}\sum_j \beta_{ij}\psi_j\left(\frac{x}{x-1}\right) \tag{11}$$

follows from another well known property of hypergeometric functions [10]. All of the β_{ij} vanish except β_{15}, β_{51}, β_{36}, β_{63}, β_{22} and β_{44}, which have modulus $|\beta_{ij}|=1$. Thus from equations (10) and (11), we conclude

$$A_{11}=A_{55} \qquad A_{33}=A_{66}. \tag{12}$$

Finally we require that $\langle\sigma(1)\bar\sigma(2)\sigma(3)\bar\sigma(4)\rangle$ approaches $\langle\sigma(1)\bar\sigma(2)\rangle\langle\sigma(3)\bar\sigma(4)\rangle = |z_{12}z_{34}|^{-4/15}$ in the limit $z_{13}\to\infty$ with z_{12} and z_{34} fixed and approaches $\langle\sigma(1)\sigma(3)\rangle\langle\bar\sigma(2)\bar\sigma(4)\rangle=0$ in the limit $z_{12}\to\infty$ with z_{13} and z_{24} fixed. These two limits correspond to $x\to1$ and $x\to0$, respectively. Making use of equations (5) and (8), we obtain

$$A_{11}+A_{55}=0 \qquad B_{11}+B_{55}=1. \tag{13}$$

Equations (7), (9), (12) and (13) determine all the A_{ij} and B_{ij}. The bulk 4-spin correlation function is given by equations (5) and (8), where

$$A_{ij} = 4\frac{b^2}{a}\,\delta_{i2}\delta_{j2}+\tfrac{1}{9}\delta_{i4}\delta_{j4} \qquad B_{ij}=\delta_{i5}\delta_{j5}+\frac{b^2}{a}\,\delta_{i6}\delta_{j6}. \tag{14}$$

This result agrees with the 4-spin correlation function of the self-dual Z_N model, calculated by Zamolodchikov and Fateev [8], in the limit $N\to3$ corresponding to the 3-state Potts model.

Next we consider the semi-infinite geometry. Cardy [3] has shown that the n-point function of a critical system defined on the half plane $y>0$ with a conformally-invariant boundary condition satisfies the same differential equations in the variables $z_1, \bar z_1, \ldots, z_n, \bar z_n$ as the bulk $2n$-point correlation function in the variables $z_1, z_2, \ldots, z_{2n}$. Thus, on making the replacement $z_1, z_2, z_3, z_4 \to z_1, \bar z_1, z_2, \bar z_2$ in equations (3), (5) and (6), we obtain

$$\begin{aligned}\langle\sigma(1)\bar\sigma(2)\rangle_{\text{half space}} &= r^{-4/15}\sum_i A_i\psi_i\left(\frac{4y_1y_2}{r^2+4y_1y_2}\right)\\ &= (4y_1y_2)^{-2/15}\sum_i B_i\psi_i\left(\frac{r^2}{r^2+4y_1y_2}\right)\end{aligned} \tag{15}$$

for the spin–spin correlation function. Here $r=|z_1-z_2|$ and

$$B_j=\sum_i A_i\alpha_{ij} \tag{16}$$

as follows from equation (6). We now determine the coefficients A_i, B_i.

The results (8) and (14) for the bulk 4-point function imply the operator product expansions

$$\begin{aligned}\sigma(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{r})\bar{\sigma}(\boldsymbol{R}-\tfrac{1}{2}\boldsymbol{r})&\approx r^{-4/15}[1+Cr^{4/5}\varepsilon(\boldsymbol{R})+\ldots]\\ \sigma(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{r})\sigma(\boldsymbol{R}-\tfrac{1}{2}\boldsymbol{r})&\approx r^{-4/15}[Dr^{2/15}\bar{\sigma}(\boldsymbol{R})+Er^{4/3}\omega(\boldsymbol{R})+\ldots]\end{aligned} \tag{17}$$

in the limit $\boldsymbol{r}\to 0$. Here ε is the principal energy operator, and ω is a non-leading magnetization operator. The quantities C, D, E are structure constants. The dots denote contributions of operators with scaling dimensions that differ from those of the indicated operators by integers. For consistency with the first of equations (17) in the limit $\boldsymbol{r}\to 0$, the coefficients B_i in equation (15) must satisfy

$$B_1+B_5=1 \qquad B_2=B_4=0 \tag{18}$$

which with equations (7) and (16) yields

$$A_5=A_6=0. \tag{19}$$

First we consider the case of free boundary conditions. According to a result of Cardy [3] the spin–spin correlation function decays as $r^{-4/3}$ in the surface limit $r\to\infty$ with y_1 and y_2 fixed. Together with equation (19) this requires that all the A_i except A_4 vanish. Equations (7), (16), (18), and (19) determine A_4 uniquely and yield

$$\langle\sigma(1)\bar{\sigma}(2)\rangle_{\text{free}}=\tfrac{1}{6}(\sqrt{5}+1)r^{-4/15}\psi_4\left(\frac{4y_1y_2}{r^2+4y_1y_2}\right). \tag{20}$$

Next we consider the case of boundary spins fixed in the state $\sigma=1$. With this boundary condition the profiles $\langle\sigma\rangle=\langle\bar{\sigma}\rangle\propto y^{-2/15}$ are non-vanishing, which together with equation (19) requires $A_1\neq 0$ in equation (15). In general the connected part of the spin–spin correlation function decays more rapidly parallel to the surface for fixed spin boundary conditions than for free boundary conditions [3, 6, 11]. Thus all the A_i vanish except A_1, which is uniquely determined by equations (7), (16) and (18). In this way we obtain

$$\langle\sigma(1)\bar{\sigma}(2)\rangle_{\text{fixed}}=\tfrac{1}{2}(\sqrt{5}+1)r^{-4/15}\psi_1\left(\frac{4y_1y_2}{r^2+4y_1y_2}\right). \tag{21}$$

For fixed-spin boundary conditions there is a second non-vanishing spin–spin correlation function $\langle\sigma\sigma\rangle_{\text{fixed}}$. It can also be expanded in terms of the functions ψ_i as in equation (15), with different coefficients $\tilde{A}_i$, $\tilde{B}_i$. For consistency with the second operator product expansion in (17), all the $\tilde{B}_i$ except $\tilde{B}_2$ and $\tilde{B}_4$ must vanish. Again we assume that $\langle\sigma\sigma\rangle_{\text{fixed}}$ decays faster than $\langle\sigma\bar{\sigma}\rangle_{\text{free}}$ parallel to the boundary. These considerations, the normalization $\langle\sigma(1)\rangle\langle\sigma(2)\rangle=\frac{1}{2}(\sqrt{5}+1)(4y_1y_2)^{-2/15}$ fixed by equation (21), and equations (7) and (16) then determine all the $\tilde{A}_i$ and $\tilde{B}_i$. The final result is

$$\langle\sigma(1)\sigma(2)\rangle_{\text{fixed}}=\tfrac{1}{2}(\sqrt{5}+1)r^{-4/15}\psi_5\left(\frac{4y_1y_2}{r^2+4y_1y_2}\right). \tag{22}$$

According to Burkhardt and Cardy [11], in a d-dimensional semi-infinite critical system the connected pair correlation function $\langle\phi(1)\phi(2)\rangle-\langle\phi(1)\rangle\langle\phi(2)\rangle$ of any operator with a non-vanishing profile $\langle\phi\rangle$ decays, in general, as r^{-2d} parallel to the boundary. Our results (21) and (22) for the spin–spin correlation function with fixed-spin boundary conditions are consistent with this prediction. The prediction does not apply to the two-point correlations of the quantity $\sigma-\bar{\sigma}$ in the presence of boundary spins fixed in state 1 since the profile $\langle\sigma-\bar{\sigma}\rangle$ vanishes identically. From equations (21) and (22) we see that $\langle[\sigma(1)-\bar{\sigma}(1)][\sigma(2)-\bar{\sigma}(2)]\rangle_{\text{fixed}}$ decays as r^{-6} in the surface limit $r\to\infty$ with y_1 and y_2 fixed. The corresponding surface scaling dimension 3 has also been derived by Cardy [12] for the 3-state Potts model.

In equations (8) and (14) the bulk 4-spin correlation is expressed in terms of only two solutions, $\psi_2(x)$ and $\psi_4(x)$, of the sixth-order differential equation (4). It is an interesting fact that one can generate the other four solutions from these two solutions without using the detailed form of the differential equation. If a function $(z_{12}z_{34})^{-2\Delta}\psi(x)$ satisfies the differential equations for the bulk 4-spin correlation function, then $(z_{12}z_{34})^{-2\Delta}\psi(x^{-1})$ and $(z_{12}z_{34})^{-2\Delta}(1-x^{-1})^{2\Delta}\psi(1-x)$ are also solutions. This follows from the symmetry of the differential equations in the coordinates $z_1,\ldots,z_4$. Thus from each of the solutions $\psi_2(x)$, $\psi_4(x)$ we obtain two additional solutions. Rewriting the hypergeometric functions with arguments x^{-1} and $1-x$ in terms of hypergeometric functions with argument x, we reproduce the other four solutions in equation (5).

In the self-dual Z_N model [8] the spin variables σ take the N values $1,\omega,\omega^2,\ldots,\omega^{N-1}$, where $\omega=\exp(2\pi\mathrm{i}/N)$. The scaling indices $\Delta(N,k)$ of the magnetization operators σ^k, $k=1,2,\ldots,N$ are given by

$$\Delta(N,k)=\frac{k(N-k)}{2N(N+2)} \tag{23}$$

and the central charge by $c=2(N-1)/(N+2)$. Note that $c>1$ for $N>4$. Zamolodchikov and Fateev [8] have calculated the bulk correlation function $\langle\sigma(1)\bar{\sigma}(2)\sigma(3)^k\bar{\sigma}(4)^k\rangle$ of the system. As in equations (8) and (14), the correlation function is given in terms of two functions $\psi_2^{(N,k)}(x)$ and $\psi_4^{(N,k)}(x)$, which, apart from power-law prefactors, are hypergeometric.

Specializing to the case $k=1$, we generate two additional solutions from each of these functions by the procedure outlined two paragraphs above. Constructing surface correlation functions from the set of six solutions, we obtain

$$\begin{aligned}
\langle\sigma(1)\bar{\sigma}(2)\rangle_{\text{free}}&=\frac{A}{N}r^{-4\Delta(N,1)}\xi^{(N^2-1)/N(N+2)}F\left(\tfrac{N+1}{N+2},\tfrac{N-1}{N+2};\tfrac{2N+2}{N+2};\xi\right)\\
\langle\sigma(1)\bar{\sigma}(2)\rangle_{\text{fixed}}&=Ar^{-4\Delta(N,1)}\xi^{-(N-1)/N(N+2)}F\left(\tfrac{N-1}{N+2},\tfrac{-1}{N+2};\tfrac{N}{N+2};\xi\right)\\
\langle\sigma(1)\sigma(2)\rangle_{\text{fixed}}&=Ar^{-4\Delta(N,1)}\xi^{-(N-1)/N(N+2)}(1-\xi)^{(N-2)/N(N+2)}F\left(\tfrac{1}{N+2},\tfrac{-1}{N+2};\tfrac{N}{N+2};\xi\right)
\end{aligned} \tag{24}$$

where

$$\xi=\frac{4y_1y_2}{r^2+4y_1y_2}\qquad A=2\cos\frac{\pi}{N+2}.$$

Equations (24) are consistent with the operator–product expansion deduced from the bulk 4-spin correlation function of the Z_N model. In the limits $N \to 3$ and $N \to 2$ they reproduce our expressions (20)–(22) for the semi-infinite 3-state Potts model and Cardy's results [3] for the semi-infinite Ising model, respectively.

In the surface limit $r \to \infty$ with y_1 and y_2 fixed, the three spin–spin correlation functions in equation (24) decay as $r^{-2(1-N^{-1})}$, r^{-4} and r^{-4} respectively. The result for free boundary conditions agrees with the conjecture $x_s(N, k) = k(1 - kN^{-1})$ of Vanderzande [13] and Alcaraz [14] for the surface scaling dimension of the operator σ^k. The r^{-4} decay for fixed boundary spins is consistent with the general result r^{-2d} for operators with non-vanishing profiles given in [11]. From equation (24) one finds that $\langle[\sigma(1) - \bar{\sigma}(1)][\sigma(2) - \bar{\sigma}(2)]\rangle_{\text{fixed}}$ falls off as r^{-6} parallel to the boundary.

In conclusion we have calculated the spin–spin correlation function of the critical semi-infinite 3-state Potts model with free- and fixed-spin boundary conditions. This was done by solving a sixth-order differential equation from the conformal theory. Making use of the results of Zamolodchikov and Fateev [8] for the bulk 4-spin correlation function of the critical self-dual Z_N model, we also obtained the spin–spin correlation function in the half space for arbitrary N.

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